1 N20 Paper 2 #5

(a) Assuming the Maclaurin series for $\cos x$ and $\ln(1+x)$, show that the Maclaurin series for $\cos(\ln(1+x))$ is [4]

$$1 - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{12}x^4 + \dots$$

(b) By differentiating the series in part (a), show that the Maclaurin series for $\sin(\ln(1+x))$ is [4]

$$x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

(c) Hence determine the Maclaurin series for $\tan(\ln(1+x))$ as far as the term x^3. [5]

Guideline

- **Part (a):** Substitute the Maclaurin series of $u = \ln(1+x)$ into the Maclaurin series of $\cos u$. Expand the powers up to x^4 and collect like terms.
- **Part (b):** Differentiate $\cos(\ln(1+x))$ using the chain rule, which gives $-\sin(\ln(1+x)) \cdot \frac{1}{1+x}$. Equate this to the derivative of the series from part (a), then multiply by $-(1+x)$ to isolate $\sin(\ln(1+x))$.
- **Part (c):** Use the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$. Divide the series for $\sin(\ln(1+x))$ by the series for $\cos(\ln(1+x))$ up to the x^3 term using polynomial long division.

Solution ....

(a) The standard Maclaurin series are:

$$\cos u = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \dots$$

$$u = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Substitute u into the cosine series:

$$\cos(\ln(1+x)) = 1 - \frac{1}{2}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\right)^2 + \frac{1}{24}\left(x - \frac{x^2}{2} + \dots\right)^4$$

Now, expand the powers, keeping only terms up to x^4: For u^2:

$$u^2 = x^2 + 2(x)\left(-\frac{x^2}{2}\right) + 2(x)\left(\frac{x^3}{3}\right) + \left(-\frac{x^2}{2}\right)^2 + \dots$$

$$u^2 = x^2 - x^3 + \left(\frac{2}{3} + \frac{1}{4}\right)x^4 = x^2 - x^3 + \frac{11}{12}x^4$$

For u^4:

$$u^4 = x^4 + \dots$$

Substitute these back into the expansion:

$$\cos(\ln(1+x)) = 1 - \frac{1}{2}\left(x^2 - x^3 + \frac{11}{12}x^4\right) + \frac{1}{24}(x^4)$$

$$= 1 - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{11}{24}x^4 + \frac{1}{24}x^4$$

$$= 1 - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{10}{24}x^4$$
$$= 1 - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{12}x^4 + \dots$$

Q.E.D.

(b) Differentiate $\cos(\ln(1+x))$ with respect to x using the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}[\cos(\ln(1+x))] = -\sin(\ln(1+x)) \cdot \frac{1}{1+x}$$

Now, differentiate the Maclaurin series from part (a):

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(1 - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{12}x^4\right) = -x + \frac{3}{2}x^2 - \frac{5}{3}x^3 + \dots$$

Equate the two derivatives:

$$-\sin(\ln(1+x)) \cdot \frac{1}{1+x} = -x + \frac{3}{2}x^2 - \frac{5}{3}x^3 + \dots$$

Multiply both sides by $-(1+x)$:

$$\sin(\ln(1+x)) = (1+x)\left(x - \frac{3}{2}x^2 + \frac{5}{3}x^3\right)$$

Expand the brackets up to x^3:

$$\sin(\ln(1+x)) = x - \frac{3}{2}x^2 + \frac{5}{3}x^3 + x^2 - \frac{3}{2}x^3 + \dots$$
$$= x + \left(-\frac{3}{2} + 1\right)x^2 + \left(\frac{5}{3} - \frac{3}{2}\right)x^3$$
$$= x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

Q.E.D.

(c) We use the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

$$\tan(\ln(1+x)) = \frac{x - \frac{1}{2}x^2 + \frac{1}{6}x^3}{1 - \frac{1}{2}x^2 + \frac{1}{2}x^3}$$

Using polynomial long division (or series multiplication with the binomial expansion of the denominator):

0	0	1	$-\frac{1}{2}$	$\frac{1}{6}$
$\frac{1}{2}$		0	0	0
$-\frac{1}{2}$			0	$\frac{1}{2}$
				0
	0	1	$-\frac{1}{2}$	$\frac{2}{3}$

The quotient is the required Maclaurin series:

$$\tan(\ln(1+x)) = x - \frac{1}{2}x^2 + \frac{2}{3}x^3 + \dots$$

2 **N21 Paper 1 #11**

(a) Prove by mathematical induction that $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2e^x\right) = [x^2 + 2nx + n(n-1)]e^x$ for $n \in \mathbb{Z}^+$. [7]

(b) Hence or otherwise, determine the Maclaurin series of $f(x) = x^2e^x$ in ascending powers of x, up to and including the term in x^4. [3]

(c) Hence or otherwise, determine the value of $\lim_{x\to 0}\left[\frac{(x^2e^x - x^2)^3}{x^9}\right]$. [4]

Guideline

- **Part (a):** Prove the base case $n = 1$ using the product rule. Assume true for $n = k$. Differentiate the k-th derivative expression again using the product rule to show it matches the formula for $n = k + 1$.
- **Part (b):** Use the general Maclaurin series formula $\sum \frac{f^{(n)}(0)}{n!}x^n$. Evaluate the n-th derivative formula from part (a) at $x = 0$ for $n = 2, 3, 4$.
- **Part (c):** Substitute the Maclaurin series from part (b) into the limit expression. Expand the numerator and evaluate the limit as $x \to 0$ by looking at the leading terms.

Solution ....

(a) Let $P(n)$ be the proposition that $\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2e^x) = [x^2 + 2nx + n(n-1)]e^x$.

Base Case ($n = 1$): Using the product rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2e^x) = 2xe^x + x^2e^x = (x^2 + 2x)e^x$$

Using the formula with $n = 1$:

$$[x^2 + 2(1)x + 1(0)]e^x = (x^2 + 2x)e^x$$

Since LHS = RHS, $P(1)$ is true.

Inductive Hypothesis: Assume $P(k)$ is true for some $k \in \mathbb{Z}^+$:

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}(x^2e^x) = [x^2 + 2kx + k(k-1)]e^x$$

Inductive Step ($n = k + 1$): We differentiate the k-th derivative to find the $(k + 1)$-th derivative:

$$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}(x^2e^x) = \frac{\mathrm{d}}{\mathrm{d}x}\left([x^2 + 2kx + k(k-1)]e^x\right)$$

Apply the product rule ($u = x^2 + 2kx + k(k-1)$, $v = e^x$):

$$= (2x + 2k)e^x + [x^2 + 2kx + k(k-1)]e^x$$

Factor out e^x and group like terms:

$$= [x^2 + 2kx + 2x + k^2 - k + 2k]e^x$$

$$= [x^2 + 2(k+1)x + k^2 + k]e^x$$

$$= [x^2 + 2(k+1)x + (k+1)k]e^x$$

Notice this perfectly matches the proposition formula where n is replaced by $(k + 1)$.

Conclusion: Since $P(1)$ is true, and $P(k) \implies P(k + 1)$, then by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$.

(b) The Maclaurin series is given by $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$. We evaluate the derivatives at $x = 0$: $f(0) = (0^2)e^0 = 0$ $f'(0) = [0 + 0 + 0]e^0 = 0$ $f''(0) = [0 + 0 + 2(1)]e^0 = 2$ $f'''(0) = [0 + 0 + 3(2)]e^0 = 6$ $f^{(4)}(0) = [0 + 0 + 4(3)]e^0 = 12$

Now construct the series:

$$f(x) = 0 + 0x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 + \frac{12}{4!}x^4 + \dots$$

$$f(x) = x^2 + x^3 + \frac{1}{2}x^4 + \dots$$

(Alternatively: $x^2e^x = x^2(1 + x + \frac{x^2}{2} + \dots) = x^2 + x^3 + \frac{1}{2}x^4 + \dots$)

(c) We evaluate the limit:

$$L = \lim_{x\to 0}\left[\frac{(x^2e^x - x^2)^3}{x^9}\right]$$

Substitute the Maclaurin series into the numerator:

$$x^2e^x - x^2 = \left(x^2 + x^3 + \frac{1}{2}x^4 + \dots\right) - x^2 = x^3 + \frac{1}{2}x^4 + \dots$$

Now, cube this expression. The leading term dominates as $x \to 0$:

$$\left(x^3 + \frac{1}{2}x^4 + \dots\right)^3 = (x^3)^3 + 3(x^3)^2\left(\frac{1}{2}x^4\right) + \dots = x^9 + \frac{3}{2}x^{10} + \dots$$

Substitute back into the limit:

$$L = \lim_{x\to 0}\frac{x^9 + \frac{3}{2}x^{10} + \dots}{x^9}$$

Divide numerator and denominator by x^9:

$$L = \lim_{x\to 0}\left(1 + \frac{3}{2}x + \dots\right)$$

As $x \to 0$, all higher-order terms vanish.

$$L = 1$$

3 M21 TZ2 Paper 2 #9

(a) Write down the first three terms of the binomial expansion of $(1+t)^{-1}$ in ascending powers of t. [1]

(b) By using the Maclaurin series for $\cos x$ and the result from part (a), show that the Maclaurin series for $\sec x$ up to and including the term in x^4 is $1 + \frac{x^2}{2} + \frac{5x^4}{24}$. [4]

(c) By using the Maclaurin series for $\arctan x$ and the result from part (b), find $\lim_{x\to 0}\left(\frac{x \arctan 2x}{\sec x - 1}\right)$. [3]

Guideline

- **Part (a):** Use the standard binomial expansion formula for $(1 + x)^n$ with $n = -1$.
- **Part (b):** Use $\sec x = (\cos x)^{-1}$. Substitute the Maclaurin series of $\cos x$ into your expression from part (a) by letting $t = \cos x - 1$.
- **Part (c):** Substitute the Maclaurin series for $\arctan(2x)$ and $\sec x$ into the limit expression. Evaluate the limit by looking at the leading terms of the numerator and denominator.

Solution ....

(a) Using the binomial expansion formula for $(1+t)^n$:

$$(1+t)^{-1} = 1 + (-1)t + \frac{(-1)(-2)}{2!}t^2 + \dots$$
$$= 1 - t + t^2$$

(b) We know that $\sec x = \frac{1}{\cos x} = (\cos x)^{-1}$. The Maclaurin series for $\cos x$ is:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots$$

We can write this in the form $1+t$ where $t = -\frac{x^2}{2} + \frac{x^4}{24} - \dots$ Now, substitute this t into the expansion from part (a):

$$\sec x = (1+t)^{-1} = 1 - t + t^2 - \dots$$
$$= 1 - \left(-\frac{x^2}{2} + \frac{x^4}{24}\right) + \left(-\frac{x^2}{2} + \frac{x^4}{24}\right)^2 - \dots$$

Expand and collect terms up to x^4:

$$= 1 + \frac{x^2}{2} - \frac{x^4}{24} + \left(-\frac{x^2}{2}\right)^2 + \dots$$
$$= 1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^4}{4}$$

Get a common denominator for the x^4 terms:

$$= 1 + \frac{x^2}{2} + \frac{6x^4}{24} - \frac{x^4}{24}$$
$$= 1 + \frac{x^2}{2} + \frac{5x^4}{24}$$

Q.E.D.

(c) The standard Maclaurin series for $\arctan u = u - \frac{u^3}{3} + \dots$ Therefore, $\arctan(2x) = 2x - \frac{(2x)^3}{3} + \dots = 2x - \frac{8x^3}{3} + \dots$ Substitute this and the series for $\sec x$ into the limit:

$$L = \lim_{x\to 0} \frac{x\left(2x - \frac{8x^3}{3} + \dots\right)}{\left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \dots\right) - 1}$$

Simplify the numerator and denominator:

$$L = \lim_{x\to 0} \frac{2x^2 - \frac{8x^4}{3} + \dots}{\frac{x^2}{2} + \frac{5x^4}{24} + \dots}$$

Divide numerator and denominator by x^2:

$$L = \lim_{x\to 0} \frac{2 - \frac{8x^2}{3} + \dots}{\frac{1}{2} + \frac{5x^2}{24} + \dots}$$

As $x \to 0$, all terms with x vanish.

$$L = \frac{2}{1/2} = 4$$

4 M21 TZ1 Paper 1 #12

Let $f(x) = \sqrt{1+x}$ for $x > -1$.

(a) Show that $f''(x) = -\dfrac{1}{4\sqrt{(1+x)^3}}$. [3]

(b) Use mathematical induction to prove that $f^{(n)}(x) = \left(-\dfrac{1}{4}\right)^{n-1} \dfrac{(2n-3)!}{(n-2)!}(1+x)^{\frac{1}{2}-n}$ for $n \in \mathbb{Z}$, $n \geq 2$. [9]

Let $g(x) = e^{mx}$, $m \in \mathbb{Q}$.
Consider the function h defined by $h(x) = f(x) \times g(x)$ for $x > -1$.
It is given that the x^2 term in the Maclaurin series for $h(x)$ has a coefficient of $\dfrac{7}{4}$.

(c) Find the possible values of m. [8]

Guideline

- **Part (a):** Write the square root as a fractional exponent $(1+x)^{1/2}$ and differentiate twice using the power rule and chain rule.
- **Part (b):** Prove the base case $n = 2$. Assume the formula is true for $n = k$ and differentiate it to prove it for $n = k+1$. Be careful with the factorial manipulation: multiplying by $(2k-1)$ requires you to introduce $(2k-2)$ to properly form the $(2k-1)!$ factorial.
- **Part (c):** The Maclaurin series for $h(x)$ is the product of the Maclaurin series for $f(x)$ and $g(x)$. Find the series for each up to the x^2 term, multiply them out, and solve the resulting quadratic equation for m.

Solution ....

(a) We can rewrite $f(x)$ as $f(x) = (1+x)^{1/2}$. Differentiate using the power rule:

$$f'(x) = \frac{1}{2}(1+x)^{-1/2}$$

Differentiate again to find $f''(x)$:

$$f''(x) = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(1+x)^{-3/2}$$

$$f''(x) = -\frac{1}{4}(1+x)^{-3/2} = -\frac{1}{4\sqrt{(1+x)^3}}$$

Q.E.D.

(b) Let $P(n)$ be the proposition that $f^{(n)}(x) = \left(-\frac{1}{4}\right)^{n-1} \frac{(2n-3)!}{(n-2)!}(1+x)^{\frac{1}{2}-n}$.

Base Case ($n = 2$): LHS $= f''(x) = -\frac{1}{4}(1+x)^{-3/2}$. RHS $= \left(-\frac{1}{4}\right)^{2-1} \frac{(4-3)!}{(2-2)!}(1+x)^{\frac{1}{2}-2} = \left(-\frac{1}{4}\right)^{1} \frac{1!}{0!}(1+x)^{-3/2} = -\frac{1}{4}(1+x)^{-3/2}$. Since LHS = RHS, $P(2)$ is true.

Inductive Hypothesis: Assume $P(k)$ is true for some integer $k \geq 2$:

$$f^{(k)}(x) = \left(-\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k}$$

Inductive Step ($n = k+1$): Differentiate $f^{(k)}(x)$ to find $f^{(k+1)}(x)$:

$$f^{(k+1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k}\right]$$

$$= \left(-\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!} \left(\frac{1}{2} - k\right) (1+x)^{\frac{1}{2}-k-1}$$

Rewrite the new exponent multiplier $\left(\frac{1}{2} - k\right)$ as $\left(\frac{1-2k}{2}\right) = -\frac{2k-1}{2}$:

$$= \left(-\frac{1}{4}\right)^{k-1} \left(-\frac{2k-1}{2}\right) \frac{(2k-3)!}{(k-2)!} (1+x)^{\frac{1}{2}-(k+1)}$$

We need to match the constant to the desired formula for $(k+1)$. Notice that:

$$\left(-\frac{1}{4}\right)^{k-1} \left(-\frac{2k-1}{2}\right) = \left(-\frac{1}{4}\right)^{k} (-4) \left(-\frac{2k-1}{2}\right) = \left(-\frac{1}{4}\right)^{k} (2)(2k-1)$$

Now integrate this back with the factorials:

$$= \left(-\frac{1}{4}\right)^{k} \frac{2(2k-1)(2k-3)!}{(k-2)!} (1+x)^{\frac{1}{2}-(k+1)}$$

Multiply the numerator and denominator by $(k-1)$ to form $(k-1)!$ on the bottom:

$$= \left(-\frac{1}{4}\right)^{k} \frac{2(k-1)(2k-1)(2k-3)!}{(k-1)(k-2)!}$$

$$= \left(-\frac{1}{4}\right)^{k} \frac{(2k-2)(2k-1)(2k-3)!}{(k-1)!}$$

Notice that $(2k-1)(2k-2)(2k-3)! = (2k-1)!$, which perfectly matches $(2(k+1)-3)!$:

$$= \left(-\frac{1}{4}\right)^{k} \frac{(2k-1)!}{(k-1)!} (1+x)^{\frac{1}{2}-(k+1)}$$

This is exactly the formula for $n = k+1$.

Conclusion: Since $P(2)$ is true, and $P(k) \implies P(k+1)$, the proposition is true for all $n \in \mathbb{Z}, n \geq 2$ by mathematical induction.

(c) We need the Maclaurin series for $h(x) = f(x)g(x)$ up to the x^2 term. First, find the Maclaurin series for $f(x)$ up to x^2 using $f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2$:

$$f(0) = \sqrt{1} = 1$$

$$f'(0) = \frac{1}{2}(1)^{-1/2} = \frac{1}{2}$$

$$f''(0) = -\frac{1}{4}(1)^{-3/2} = -\frac{1}{4}$$

$$f(x) \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

Next, find the standard Maclaurin series for $g(x) = e^{mx}$:

$$g(x) \approx 1 + (mx) + \frac{(mx)^2}{2!} = 1 + mx + \frac{m^2}{2}x^2$$

Multiply the two series to find the x^2 term of $h(x)$:

$$h(x) \approx \left(1 + \frac{1}{2}x - \frac{1}{8}x^2\right) \left(1 + mx + \frac{m^2}{2}x^2\right)$$

The x^2 term is formed by taking cross-products that result in degree 2:

$$(1)\left(\frac{m^2}{2}x^2\right) + \left(\frac{1}{2}x\right)(mx) + \left(-\frac{1}{8}x^2\right)(1)$$

The coefficient of x^2 is:

$$\frac{m^2}{2} + \frac{m}{2} - \frac{1}{8}$$

We are given this coefficient equals $\frac{7}{4}$:

$$\frac{m^2}{2} + \frac{m}{2} - \frac{1}{8} = \frac{7}{4}$$

Multiply the entire equation by 8 to clear denominators:

$$4m^2 + 4m - 1 = 14$$

$$4m^2 + 4m - 15 = 0$$

Factorize the quadratic equation:

$$(2m - 3)(2m + 5) = 0$$

Therefore, the possible values for m are:

$$m = \frac{3}{2}, \quad m = -\frac{5}{2}$$

5 Specimen Paper 1 # 12

The function f is defined by $f(x) = e^{\sin x}$.

(a) Find the first two derivatives of $f(x)$ and hence find the Maclaurin series for $f(x)$ up to and including the x^2 term. [8]

(b) Show that the coefficient of x^3 in the Maclaurin series for $f(x)$ is zero. [4]

(c) Using the Maclaurin series for $\arctan x$ and $e^{3x} - 1$, find the Maclaurin series for $\arctan(e^{3x} - 1)$ up to and including the x^3 term. [6]

(d) Hence, or otherwise, find $\lim_{x \to 0} \dfrac{f(x) - 1}{\arctan(e^{3x} - 1)}$.

Guideline

- **Part (a):** Use the chain rule for the first derivative, and the product rule (along with the chain rule) for the second derivative. Evaluate $f(0), f'(0)$, and $f''(0)$ to build the Maclaurin series.
- **Part (b):** Differentiate $f''(x)$ to find $f'''(x)$. Evaluate $f'''(0)$ to show it equals 0. Alternatively, substitute the Maclaurin series for $\sin x$ into the series for e^u.
- **Part (c):** Find the series for $u = e^{3x} - 1$ up to x^3. Then substitute this expression into the Maclaurin series for $\arctan u = u - \frac{u^3}{3} + \ldots$
- **Part (d):** Substitute the series from (a), (b), and (c) into the limit. Factor out x from the numerator and denominator to evaluate the limit as $x \to 0$.

Solution ....

(a) First, find the derivatives of $f(x) = e^{\sin x}$: Using the chain rule:

$$f'(x) = (\cos x)e^{\sin x}$$

Using the product and chain rules for the second derivative:

$$f''(x) = (-\sin x)e^{\sin x} + (\cos x)(\cos x)e^{\sin x}$$

$$f''(x) = e^{\sin x}(\cos^2 x - \sin x)$$

Now, evaluate the function and its derivatives at $x = 0$:

$$f(0) = e^{\sin 0} = e^0 = 1$$

$$f'(0) = (\cos 0)e^{\sin 0} = (1)(1) = 1$$

$$f''(0) = e^{\sin 0}(\cos^2 0 - \sin 0) = (1)(1^2 - 0) = 1$$

The Maclaurin series is $f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$:

$$f(x) \approx 1 + 1x + \frac{1}{2}x^2$$

$$f(x) = 1 + x + \frac{1}{2}x^2 + \dots$$

(b) We need to find $f'''(0)$. Differentiate $f''(x) = e^{\sin x}(\cos^2 x - \sin x)$ using the product rule:

$$f'''(x) = \left[(\cos x)e^{\sin x}\right](\cos^2 x - \sin x) + e^{\sin x}\left[-2\cos x \sin x - \cos x\right]$$

Evaluate at $x = 0$:

$$f'''(0) = \left[(\cos 0)e^{\sin 0}\right](\cos^2 0 - \sin 0) + e^{\sin 0}\left[-2\cos 0 \sin 0 - \cos 0\right]$$

$$f'''(0) = [(1)(1)](1 - 0) + (1)[-0 - 1]$$

$$f'''(0) = 1(1) + 1(-1) = 1 - 1 = 0$$

The coefficient of x^3 is given by $\frac{f'''(0)}{3!} = \frac{0}{6} = 0$. Q.E.D.
(Alternative method: Substitute $\sin x = x - \frac{x^3}{6} + \dots$ *into* $e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \dots$ *and collect* x^3 *terms:* $-\frac{1}{6}x^3 + \frac{1}{6}x^3 = 0$.*)*

(c) First, find the Maclaurin series for $u = e^{3x} - 1$:

$$e^{3x} = 1 + (3x) + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \dots$$

$$e^{3x} - 1 = 3x + \frac{9x^2}{2} + \frac{27x^3}{6} + \dots = 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \dots$$

The Maclaurin series for $\arctan u$ is $u - \frac{u^3}{3} + \dots$ Substitute $u = 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3$:

$$\arctan\big(e^{3x} - 1\big) = \left(3x + \frac{9}{2}x^2 + \frac{9}{2}x^3\right) - \frac{1}{3}\left(3x + \frac{9}{2}x^2 + \dots\right)^3 + \dots$$

Expand the cubic term, keeping only up to x^3. The only relevant term from $(3x + \dots)^3$ is $(3x)^3 = 27x^3$.

$$\arctan\big(e^{3x} - 1\big) = 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 - \frac{1}{3}(27x^3)$$

$$= 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 - 9x^3$$

$$= 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3 + \dots$$

(d) We want to find $\lim_{x\to 0} \frac{f(x)-1}{\arctan(e^{3x}-1)}$. Substitute the Maclaurin series found in parts (a), (b), and (c):

$$f(x) - 1 = \left(1 + x + \frac{1}{2}x^2 + 0x^3 + \dots\right) - 1 = x + \frac{1}{2}x^2 + \dots$$

The limit becomes:

$$L = \lim_{x\to 0} \frac{x + \frac{1}{2}x^2 + \dots}{3x + \frac{9}{2}x^2 - \frac{9}{2}x^3 + \dots}$$

Divide the numerator and the denominator by the lowest power of x (which is x):

$$L = \lim_{x\to 0} \frac{1 + \frac{1}{2}x + \dots}{3 + \frac{9}{2}x - \frac{9}{2}x^2 + \dots}$$

As $x \to 0$, all terms containing x approach 0:

$$L = \frac{1}{3}$$